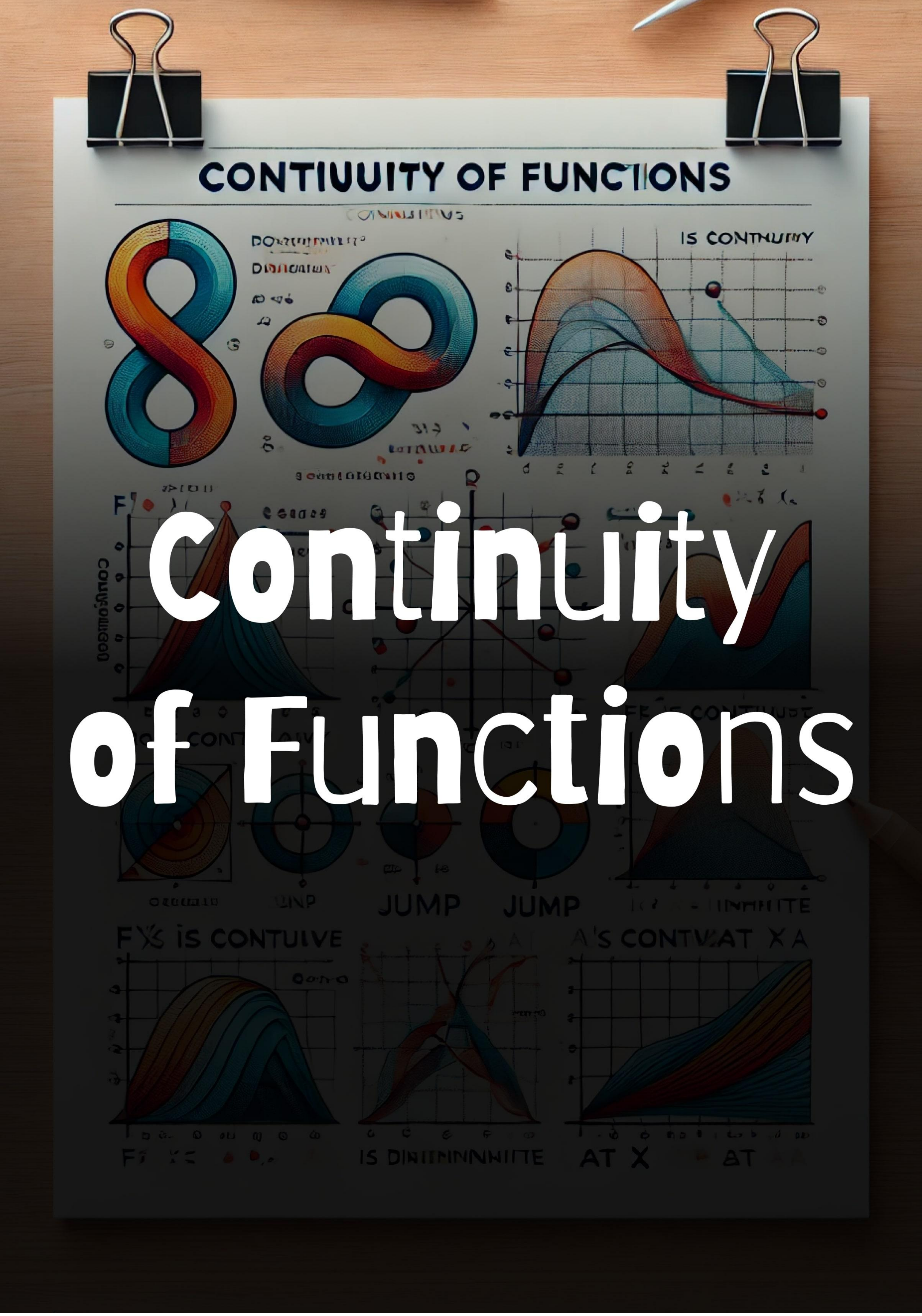
Continuity of Functions



CONTINUITY OF FUNCTIONS

If a function $f(x)$ at $x = a$ has equal results for $x = a^+, a^-, a$ then function $f(x)$ is a Continuous function.

It is continuous if -

$$\boxed{LHL = RHL = f(a)}$$

$$\boxed{\underset{x\to a^-}{Lt} f(x) = \underset{x\to a^+}{Lt} f(x) = f(a)}$$

e.g. $f(x) = [x]$ at $x = 2$

$\underset{x\to 2^-}{Lt} f(x) = 1$ $\quad$ $\underset{x\to 2^+}{Lt} f(x) = 2$ $\quad$ $f(2) = [2] = 2$

It is not continaos function

eg: $f(x) = x^2$ at $x = 2$

$\underset{x\to 2^-}{Lt} f(x) = 4$ $\quad$ $\underset{x\to 2^+}{Lt} f(x) = 4$ $\quad$ $f(2) = 4$

It is continuous function

CONTINUITY IN AN INTERVAL (a,b)

If $f(x)$ is continuous function at all points of domain $f(x)$ is continuous in (a, b).

Continuity in [a, b]

If $f(x)$ is continuous at all points in $[a, b]$ as well as at end points, then we can say $f(x)$ is continuous in $[a, b]$

For the end points $[a, b]$, we check that

$\boxed{f(a) = \underset{x\to a^+}{Lt} f(x)}$ & $\boxed{f(b) = \underset{x\to b^-}{Lt} f(x)}$

↳ and $f(a)$ $\quad$ ↳ and $f(b)$

If $f(x)$ & $g(x)$ be two continuous functions $\forall x$, then–

(i) $f(x) \pm g(x)$ is also continuous

(ii) $Kf(x)$, $K g(x)$ is also continuous

(iii) $f(x) g(x)$ is also continuous

(iv) $\dfrac{f(x)}{g(x)}$ is also continuous, $g(x) \neq 0$

- All real polynomials are continuous $\forall x \in R$, $f(x) = \sin x$, $\cos x$ are continuous $\forall x \in R$
- If $f(x)$ is continuous in $[a, b]$ and $f(a), f(b)$ are of opposite sign then $f(x)$ has at least one root between a, b
- If $f(x)$ is continuous in $[a, b]$ and $f(x)$ has range $[m, M]$ if N is any no. between m & M, then there exists as no. c between $[a, b]$ such that $f(c) = N$

Ques: $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ at $x = 0$, discuss the continuity?

Sol: At 0^+, $f(x) = \dfrac{e^{\frac{1}{h}} - 1}{e^{\frac{1}{h}} + 1}$, 0^- $f(x) = \dfrac{e^{-\frac{1}{h}} - 1}{e^{-\frac{1}{h}} + 1}$

$= \dfrac{1 - e^{1/h}}{1 + e^{-h}}$

$\therefore$ It is not continuous

Ques: Discuss the continuity if $f(x) = [x] + [-x]$ at integral points.

Sol: Let n be an integral point

$n^+ \Rightarrow n + h$

$f(x) = [n+h] + [-n-h]$ $\quad h \to 0$

$= n + [-n-1] \Rightarrow -1$

$n^- \Rightarrow n - h$

$f(x) = [n-h] + [-n+h]$

$= (n-1) + (-n) \Rightarrow -1$

$f(n) = [n] + [-n] = 0$

It is discontinuous at integral points.

Ques: If $f(x) = \begin{cases} 2a + \sin x & , -\frac{\pi}{6} < x < 0 \\ b & , x = 0 \\ 2a + b + \cos x & , x > 0 \end{cases}$ is continous at $x = 0$, then find a, b.

Sol: At $x = 0^{-}$ $\Rightarrow$ $0-h = -h$

$2a + \sin x \Rightarrow 2a + \sin(-h) = 2a$

$2a = b$

At $x = 0^{+}$ $\Rightarrow$ $0+h = +h$

$2a + b + 1 = b$

$\Rightarrow$ $\boxed{a = -\frac{1}{2} \qquad b = -1}$

Ques: If $f(x) = \dfrac{\sin 2x + a\sin x + b\cos x}{x^3}$ is continuous at $x=0$ then find a, b & $f(0)$

Sol: At $x = 0^{-}$ $\Rightarrow 0-h = -h$

$$\lim_{x\to 0^-} f(0^-) = \frac{\sin(-2h) + a\sin(-h) + b\cos(-h)}{(-h)^3} = \frac{b}{(-h)^3}$$

For limit to exist, $\frac{0}{0}$ form must be there

$\boxed{b=0}$

$$\lim_{x\to 0^+} f(x) = \frac{\sin 2h + a\sinh + b\cos h}{x^3}$$

$b=0$, $= \dfrac{\sin 2h + a\sin h}{x^3}$

$$= \frac{2\sin h\cos h + a\sin h}{x^3}$$

$$= \frac{\sin h}{h}\,\frac{(2\cos h + a)}{h^2}$$

$$= \frac{2\cos h + a}{h^2}$$

$$= \frac{a+2}{h^2}$$

$\frac{0}{0}$ must be created, $\boxed{a = -2}$

$$f(0) = \frac{2\cos h + a}{h^2}$$

$$= \frac{2(\cos h - 1)}{h^2}$$

$$= \frac{-2\sin^2 h/2}{h^2}$$

$$= \frac{2\times -2\sin^2 h/2}{\frac{h^2}{4}\times 4} \Rightarrow \boxed{-1 = f(0)}$$

Ques: If $f(x) = \begin{cases} (1+|\sin x|)^{a/|\sin x|} & ; x<0 \\ b & ; x=0 \\ e^{\frac{\tan 8x}{\tan 3x}} & ; x>0 \end{cases}$ is continuous at $x=0$, then find $a+b$.

Sol: At $0^{\ominus}$, $x=-h$ $\quad (1+|\sin -h|)^{a/|\sin x|}$

$$\Rightarrow (1+|\sin h|)^{a/|\sin h|} = e^{a}$$

$$\Rightarrow e^{a} = b$$

At $0^{\oplus}$, $e^{\frac{\tan 8h}{\tan 3h}}$ $\Rightarrow e^{\frac{\tan 8h \times 3h \times 8h}{8h \times \tan 3h \times 3h}}$

$$\Rightarrow \boxed{e^{8/3} = b}$$

$$\boxed{a = \frac{8}{3}}$$

Some Limit Questions

Ques: $\underset{x\to 0}{Lt} \left(1^{\operatorname{cosec}^2 x} + 2^{\operatorname{cosec}^2 x} + \cdots\cdots n^{\operatorname{cosec}^2 x}\right)^{\sin^2 x}$

Sol:

$$n^{\operatorname{cosec}^2 x \sin^2 x}\left(\underbrace{\left(\frac{1}{n}\right)^{\operatorname{cosec}^2 x}}_{\to 0} + \underbrace{\left(\frac{2}{n}\right)^{\operatorname{cosec}^2 x}}_{\to 0} + \cdots\cdots \left(\frac{n-1}{n}\right)^{\operatorname{cosec}^2 x} + 1\right)$$

$$\Rightarrow n(1) = n$$

$\underset{x\to 0}{\text{Lim}} \left[\frac{\sin x}{x}\right] = 0 \qquad \underset{x\to 0^+}{\text{Lim}} \left[\frac{\sin x}{x}\right] = 1$

Ques: $\underset{n\to\infty}{Lt} \cos(\pi\sqrt{n^2+n}),\ n \in I$

Sol: $\cos(\pi\sqrt{n^2+n}) = (-1)^n \cos(n\pi - \pi\sqrt{n^2+n})$

$$= (-1)^n \cos \pi(n-\sqrt{n^2+n})$$

$$= (-1)^n \cos \pi \frac{(n-\sqrt{n^2+n})}{(n+\sqrt{n^2+n})}(n+\sqrt{n^2+n})$$

$$= (-1)^n \cos \frac{\pi(-n)}{n+n\sqrt{1+\frac{1}{n}}}$$

$$= (-1)^n \cos(-\pi/2)$$

$$= 0$$

Ques: $\underset{x\to\infty}{Lt} \left(\frac{x^2-1}{x+2} - ax - b\right) = 2$, $a, b = ?$

Sol: $\underset{x\to\infty}{Lt} \dfrac{x^2-1-ax(x+2)-b(x+2)}{x+2} = 2$

$$= \frac{x^2 - 1 - ax^2 - 2ax - bx - 2b}{x+2}$$

$$= \frac{x^2(1-a) - (2a+b)x - 2b - 1}{x+2}$$

$$= 2$$

$x \to \infty$, $1 - a = 0 \Rightarrow \boxed{a = 1}$

$\Rightarrow -(2a+b) = 2 \Rightarrow \boxed{b = -4}$

Ques: $\lim_{x\to 0^+} \left(\lim_{n\to\infty} \frac{[1^2 x^x] + [2^2 x^x] + \dots + [n^2 x^x]}{n^3} \right)$

Sol:

$$1^2 x^x - 1 < [1^2 x^x] \le 1^2 x^x$$

$$2^2 x^x - 1 < [2^2 x^x] \le 2^2 x^x$$

$$\vdots$$

$$n^2 x^x - 1 < [n^2 x^x] \le n^2 x^x$$

$$\frac{1^2 x^x - 1 + 2^2 x^x - 1 \dots + n^2 x^x - 1}{n^3} < \frac{[1^2 x^x] + \dots [n^2 x^x]}{n^3} \le \frac{1^2 x^x + 2^2 x^x \dots n^2 x^x}{n^3}$$

$$\Rightarrow \frac{x^x (n)(n+1)(2n+1)}{6n^3} - \frac{n}{n^3} < \frac{[1^2 x^x] + \dots [n^2 x^x]}{n^3} \le x^x \times \frac{n(n+1)(2n+1)}{6n^3}$$

$$\Rightarrow \lim_{n\to\infty} \frac{[1^2 x^x] + \dots + [n^2 x^x]}{n^3} = \frac{x^x}{3}$$

$$\Rightarrow \frac{1}{3} \lim_{x\to 0^+} x^x \Rightarrow \frac{1}{3} \lim_{x\to 0^+} e^{x \log_e x} = \frac{1}{3} \lim_{x\to 0^+} \frac{\log x}{1/x}$$

$$\Rightarrow \frac{1}{3} \lim_{x\to 0^+} e^{\frac{1/x}{-1/x^2}}$$

$$\Rightarrow \frac{1}{3} \lim_{x\to 0^+} e^{-x} = \boxed{+1/3}$$

A function in the form of –

$$f(x) = (a - x^n)^{1/n}$$

is self reversible, i.e. $fof = x$

DISCONTINUOUS FUNCTION

A function which is not continuous at $x = -a$ is said to be discontinuous at $x = -a$.

Discontinuity may arise because of following reasons–

(1) $\lim_{x\to a^-} = \lim_{x\to a^+}$ exists & finite but not equal to $f(a)$

$$LHL = RHL \neq f(a)$$

(2) $LHL \neq RHL$

(3) LHL or RHL or both does not exist (i.e. RHL or LHL $\to \pm\infty$ or oscillate b/w two finite limits)

TYPES OF DISCONTINUITY

1. Removal discontinuity
2. First kind discontinuity (jump discontinuity)
3. Second kind discontinuity

Removal Discontinuity

If LHL = RHL both equal but not equal to $f(a)$ i.e. $LHL = RHL \neq f(a)$

eg. $f(x) = \begin{cases} 2x-1 & , x<0 \\ 2 & , x=0 \\ x^2-1 & , x>0 \end{cases}$

$LHL = -1$

$RHL = -1$

$f(0) = 2$

First Kind Discontinuity / Jump discontinuity

LHL & RHL both exist but are not equal $LHL \neq RHL$

$$\boxed{\text{Jump} = |RHL - LHL|}$$

This type of discontinuity cannot be removed

Ques: $f(x) = [x]$ at $x = 2$

Sol: 2^-, $f(x) = 1 = LHL$

2^+, $f(x) = 2 = RHL$

Jump $= 2 - 1 = 1$

Second Kind Discontinuity

Either LHL or RHL or both do not exist.

LHL or RHL $\longrightarrow \pm\infty$

LHL or RHL $\longrightarrow$ Oscillate between two finite limits eg. Let $f(x) = \frac{1}{1+e^{1/x}}$. Check continuity at $x=0$

0^-, $f(x) = \frac{1}{1+e^{-\infty}} \Rightarrow \frac{1}{1+1/e^{\infty}} = \frac{1}{1+0} = 1$

0^+, $f(x) = \frac{1}{1+e^{1/0^+}} \Rightarrow \frac{1}{1+e^{\infty}} = \frac{1}{\infty} = 0$

0, $f(x) = \frac{1}{1+e^{1/0}} \Rightarrow \frac{1}{1+e^{\infty}} = \frac{1}{\infty} = 0$

It is discontinuous of first kind   Jump $= +1$

eg. $f(x) = \frac{1}{x}$ at $x=0$

0^-, $f(x) = \frac{1}{0^{\ominus}} = -\infty$

0^+, $f(x) = \frac{1}{0^{\oplus}} = +\infty$

It is discontinuous of second kind

eg. Let $f(x) = \begin{cases} [\cos \pi x] & , 0 \le x \le 1 \\ [2x-3] & , 1 < x \le 3 \end{cases}$, find points of discontinuity.

when, $x \in (0, \frac{1}{2}]$ $x=0$

$\pi x \in (0, \frac{\pi}{2}]$ $[\cos \pi x] = 1$

$[\cos \pi x] = 0$

$x \in (\frac{1}{2}, 1]$

$\pi x \in (\frac{\pi}{2}, \pi]$ Hence discontinuous at $1/2$

$[\cos \pi x] = -1$

$2x-3=0 \Rightarrow x = \frac{3}{2}$

$x \in [1, \frac{3}{2}]$ $|2x-3| = 3-2x$

$x \in [\frac{3}{2}, 3]$ $|2x-3| = 2x-3$

$$f(x) = \begin{cases} 1 & , x=0 \\ 0 & , x \in (0, 1/2] \\ -1 & , x \in (1/2, 1] \\ 3-2x & , x \in (1, 3/2) \\ 2x-3 & , x \in [3/2, 1] \end{cases}$$

Discontinuous at $x = \left(0, \frac{1}{2}, 1\right)$

Ques: Test continuity of $f(x) = \left(\frac{x^2-1}{x-1}\right)$ at $x \to 1$

Ans: $f(x) = (x+1)$ for $x \neq 1$

$(1+h) \to (2)$, $(1-h) \to 2$, but at $f(1)$ not defined

It is discontinuous